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# ON NEWFORMS FOR KOHNEN PLUS SPACES

MASARU UEDA AND SHUNSUKE YAMANA

**ABSTRACT.** In this article, we investigate the plus space of level  $N$ , where  $4^{-1}N$  is a square-free (not necessarily odd) integer. This is a generalization of Kohnen's work. We define a Hecke isomorphism  $\wp_k$  of  $M_{k+1/2}(4M)$  onto  $M_{k+1/2}^+(8M)$  for any odd positive integer  $M$ . The methods of the proof of the newform theory are this isomorphism, Waldspurger's theorem, and the dimension identity.

## Introduction

The purpose of this paper is to establish the theory of newforms for the Kohnen plus space with respect to  $\Gamma_0(N)$ , where  $4^{-1}N$  is a square-free integer. This is a continuation of Kohnen's work (cf. [1]) in the case when  $4^{-1}N$  is odd square-free.

Let us describe our results. The space of cusp forms of weight  $k+1/2$  with respect to  $\Gamma_0(N)$  is denoted by  $S_{k+1/2}(N)$  and the Kohnen plus space  $S_{k+1/2}^+(N)$  is defined by

$$S_{k+1/2}^+(N) = \left\{ g \in S_{k+1/2}(N) \mid g(\tau) = \sum_{n \in \mathbb{N}, (-1)^k n \equiv 0,1 \pmod{4}} a_n(g) q^n \right\},$$

where  $q = e^{2\pi\sqrt{-1}\tau}$ . The  $\mathbb{C}$ -linear map  $\wp_k$  on formal power series defined by

$$\sum_{n \in \mathbb{N} \cup \{0\}} a_n q^n | \wp_k = \sum_{n \in \mathbb{N} \cup \{0\}, (-1)^k n \equiv 0,1 \pmod{4}} a_n q^n.$$

gives a Hecke equivalent isomorphism of  $S_{k+1/2}(N)$  onto  $S_{k+1/2}^+(2N)$  (see Proposition 3.3) if  $N$  is exactly divisible by 4. We define an operator  $U(d)$  by

$$\sum_n a_n q^n | U(d) = \sum_n a_{dn} q^n$$

for each positive integer  $d$ , and put  $U_k(d) = U(d) \wp_k$ .

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We define the space of newforms  $S_{k+1/2}^{\text{new},+}(N)$  for  $S_{k+1/2}^+(N)$  to be the orthogonal complement of

$$\sum_{p|4^{-1}N} \left( S_{k+1/2}^+(p^{-1}N) + S_{k+1/2}^+(p^{-1}N)|U_k(p^2) \right),$$

where  $p$  extends over all prime divisors of  $4^{-1}N$ , in  $S_{k+1/2}^+(N)$  with respect to the Petersson inner product. Let  $\tilde{T}(p^2)$  (resp.  $T(p)$ ) be the usual Hecke operator on the space of modular forms of half-integral (resp. integral) weight. The space of newforms for  $S_{2k}(\Gamma_0(4^{-1}N))$  is denoted by  $S_{2k}^{\text{new}}(4^{-1}N)$ .

We shall show the following:

**Theorem.** *Suppose that  $k$  is positive and  $4^{-1}N$  is square-free.*

- (1)  $S_{k+1/2}^+(N) = \bigoplus_{a,d \geq 1, ad|4^{-1}N} S_{k+1/2}^{\text{new},+}(4a)|U_k(d^2)$ .
- (2) *The operators  $\tilde{T}(p^2)$  and  $U_k(q^2)$ , where  $(p, 4^{-1}N) = 1$  and  $q|4^{-1}N$ , fix  $S_{k+1/2}^{\text{new},+}(N)$ . Moreover,  $S_{k+1/2}^{\text{new},+}(N)$  has an orthogonal  $\mathbb{C}$ -basis which consists of common Hecke eigenforms of these operators.*
- (3) *There is a bijective correspondence, up to scalar multiple, between Hecke eigenforms in  $S_{2k}^{\text{new}}(4^{-1}N)$  and those in  $S_{k+1/2}^{\text{new},+}(N)$  in the following way. If  $\phi \in S_{2k}^{\text{new}}(4^{-1}N)$  is a primitive form, i.e.,*

$$\phi|T(p) = \omega_p \phi, \quad \phi|U(q) = \omega_q \phi$$

*for every prime number  $p \nmid 4^{-1}N$  and prime divisor  $q$  of  $4^{-1}N$ , then there is a non-zero Hecke eigenform  $g \in S_{k+1/2}^{\text{new},+}(N)$  such that*

$$g|\tilde{T}(p^2) = \omega_p g, \quad g|U_k(q^2) = \omega_q g$$

*for every prime number  $p \nmid 4^{-1}N$  and prime divisor  $q$  of  $4^{-1}N$ .*

**Remark.** *We can also establish the newform theory for a quadratic non-trivial character  $\chi$ . See Remark 5.1 in §5.*

Let us explain the contents of each section. In §1 we introduce the notion of modular forms of half-integral weight. Section 2 is an introduction to the theory of the Kohnen plus space. In §3 we describe the link between the full space of modular forms of half-integral weight and the Kohnen plus space. In §4 we describe Waldspurger's result, which we use in the proof of main results in §5.

## 1. Preliminaries

If  $z \in \mathbb{C}$  and  $\ell \in \mathbb{Z}$ , then let  $z^{1/2}$  be the square root of  $z$  such that  $-\pi/2 < \arg z^{1/2} \leq \pi/2$ , and put  $z^{\ell/2} = (z^{1/2})^\ell$ . Fix an integer  $k$ . The set  $\mathfrak{G}$  consists of all pairs  $(\gamma, \phi(\tau))$ , where  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is an element of the connected component  $\mathrm{GL}_2^+(\mathbb{R})$  of  $\mathrm{GL}_2(\mathbb{R})$  and  $\phi$  is a holomorphic function on the upper half-plane  $\mathfrak{H}$  satisfying

$$|\phi(\tau)| = (\det \gamma)^{-k/2-1/4} |c\tau + d|^{k+1/2}.$$

We define the group law of  $\mathfrak{G}$  by

$$(\gamma_1, \phi_1(\tau)) \cdot (\gamma_2, \phi_2(\tau)) = (\gamma_1 \gamma_2, \phi_1(\gamma_2 \tau) \phi_2(\tau)).$$

For a function  $h$  on  $\mathfrak{H}$  and  $\alpha = (\gamma, \phi(\tau)) \in \mathfrak{G}$ , we put  $h|\alpha(\tau) = \phi(\tau)^{-1} h(\gamma\tau)$ .

There exists an injective homomorphism  $\Gamma_0(4) \rightarrow \mathfrak{G}$  given by

$$\gamma \mapsto \gamma^* = (\gamma, j(\gamma, \tau)^{2k+1}), \quad j(\gamma, \tau) = \left(\frac{c}{d}\right) \epsilon_d^{-1} (c\tau + d)^{1/2}$$

for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ . Here  $\left(\frac{c}{d}\right)$  is the Kronecker symbol (see [2]) and

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}. \\ \sqrt{-1} & \text{if } d \equiv 3 \pmod{4}. \end{cases}$$

Fix an integer  $N$  and an even Dirichlet character  $\chi \pmod{N}$  such that  $\chi^2 = 1$ . Put  $\chi(\gamma) = \chi(d)$  for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ . We write  $N = 2^e M$ , where  $M$  is an odd integer. The 2-primary component of  $\chi$  is denoted by  $\chi_2$ . Throughout this paper (except for §4) we suppose the following conditions:

- (I)  $e$  equals either 2 or 3;
- (II) the conductor of  $\chi_2$  equals either 1 or 4.

*Remark 1.1.* We can define the Kohnen plus space  $M_{k+1/2}^+(N, \chi)$  without the condition (II). However,  $M_{k+1/2}^+(N, \chi) = \{0\}$  unless the conductor of  $\chi_2$  equals either 1 or 4.

We call a holomorphic function  $h$  on  $\mathfrak{H}$  a modular (resp. cusp) form of weight  $k + 1/2$  with respect to  $\Gamma_0(N)$  and  $\chi$  if  $h|\gamma^* = \chi(\gamma)h$  for every  $\gamma \in \Gamma_0(N)$  and it is holomorphic (resp. vanishes) at all cusps. The space of modular (resp. cusp) forms of weight  $k + 1/2$  with respect to  $\Gamma_0(N)$  and  $\chi$  is denoted by  $M_{k+1/2}(N, \chi)$  (resp.  $S_{k+1/2}(N, \chi)$ ).

We denote the  $n$ -th Fourier coefficient of  $h$  by  $a_n(h)$ . Put

$$\mathfrak{D}_{k, \chi} = \{m \in \mathbb{N} \cup \{0\} \mid \chi_2(-1)(-1)^k m \equiv 0, 1 \pmod{4}\}.$$

**Definition 1.2.** The space  $M_{k+1/2}^+(N, \chi)$  consists of all functions  $g \in M_{k+1/2}(N, \chi)$  such that  $a_n(g) = 0$  for all  $n \notin \mathfrak{D}_{k, \chi}$ . Put  $S_{k+1/2}^+(N, \chi) = M_{k+1/2}^+(N, \chi) \cap S_{k+1/2}(N, \chi)$ .

When  $\chi$  is the trivial character, we write

$$\begin{aligned} M_{k+1/2}(N) &= M_{k+1/2}(N, \chi), & S_{k+1/2}(N) &= S_{k+1/2}(N, \chi), \\ M_{k+1/2}^+(N) &= M_{k+1/2}^+(N, \chi), & S_{k+1/2}^+(N) &= S_{k+1/2}^+(N, \chi). \end{aligned}$$

For an element  $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$  and a function  $h$  on  $\mathfrak{H}$ , we put

$$h|_{k+1/2}\alpha(\tau) = (\det \alpha)^{k/2+1/4} (c\tau + d)^{-k-1/2} h(\alpha\tau).$$

We define  $\tilde{\delta}_d \in \mathfrak{G}$  and an operator  $U(d)$  on formal power series by

$$\begin{aligned} \tilde{\delta}_d &= \left( \begin{pmatrix} d & \\ & 1 \end{pmatrix}, d^{-k/2-1/4} \right), \\ \sum_{n \in \mathbb{N} \cup \{0\}} a_n q^n |U(d) &= \sum_{n \in \mathbb{N} \cup \{0\}} a_{dn} q^n \end{aligned}$$

for a positive integer  $d$ . For each positive divisor  $Q$  of  $M$  such that  $Q$  and  $M/Q$  are coprime, we choose an element  $\gamma_Q \in \mathrm{SL}_2(\mathbb{Z})$  such that

$$\gamma_Q \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & (\text{mod } Q^2), \\ \mathbf{1}_2 & (\text{mod } (Q^{-1}N)^2). \end{cases}$$

We define operators  $\tilde{W}(Q)$  and  $\tilde{Y}(Q)$  on  $M_{k+1/2}(N)$  by

$$\begin{aligned} \tilde{W}(Q) &= \gamma_Q^* \tilde{\delta}_Q, \\ \tilde{Y}(Q) &= Q^{-k/2+3/4} U(Q) \tilde{W}(Q). \end{aligned}$$

Put

$$\begin{aligned} \tau(N) &= \left( \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, (N^{1/4}(-\sqrt{-1}\tau)^{1/2})^{2k+1} \right) \in \mathfrak{G}, \\ \tilde{Y}(2^e) &= 2^{-(k/2-3/4)e} U(2^e) \tilde{W}(M) \tau(N). \end{aligned}$$

Note that  $\tau(N)^2 = 1$  and

$$(1.1) \quad \tilde{W}(Q)^2 = \epsilon_Q^{-2k-1} \chi_Q(-1) \chi_{N/Q}(Q)$$

on  $M_{k+1/2}(N)$ , where  $\chi_Q$  and  $\chi_{N/Q}$  are the  $Q$  and  $N/Q$ -primary components of  $\chi$  respectively (cf. [5, Proposition 1.18]).

The Petersson inner product  $\langle g, h \rangle$  is defined by

$$\langle g, h \rangle = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \int_{\Gamma_0(N) \backslash \mathfrak{H}} g(\tau) \overline{h(\tau)} y^{k-3/2} dx dy,$$

where  $g \in S_{k+1/2}(N, \chi)$ ,  $h \in M_{k+1/2}(N, \chi)$  and  $\tau = x + \sqrt{-1}y$ .

**Proposition 1.3.** *Let  $m$  be a positive divisor of  $N$ . Suppose that a complex valued function  $h$  on  $\mathfrak{H}$  satisfies the following conditions:*

- (i)  $h(\tau + 1) = h(\tau)$  for  $\tau \in \mathfrak{H}$ ;
- (ii)  $h|_{\tilde{\delta}_m} \in M_{k+1/2}(N, \chi)$ .

*Then the following assertions hold.*

- (1) *If one of the conditions  $m\mathfrak{f}(\chi') \nmid N$  and  $4m \nmid N$  is satisfied, then  $h = 0$ .*
- (2) *If  $N$  is divisible by  $m\mathfrak{f}(\chi')$  and  $4m$ , then  $h \in M_{k+1/2}(m^{-1}N, \chi')$ .*

*Here  $\chi' = \chi\left(\frac{m}{\cdot}\right)$  and  $\mathfrak{f}(\chi')$  is the conductor of  $\chi'$ .*

*Proof.* See [3, Lemma 7]. □

**Proposition 1.4.** *Let  $p$  be a prime divisor of  $M$  with  $\text{ord}_p M = 1$ . Suppose that  $\chi_p = 1$ . Then the operator  $\epsilon_p^{2k+1} p^{-1/2} \tilde{Y}(p)$  is an involution on  $M_{k+1/2}(N, \chi)$ . Let  $h \in M_{k+1/2}(N, \chi)$  and  $\epsilon$  either 1 or  $-1$ . Then the following conditions are equivalent:*

- (i)  $\epsilon_p^{2k+1} p^{-1/2} h|_{\tilde{Y}(p)} = \epsilon h$ ;
- (ii)  $a_n(h) = 0$  if  $\left(\frac{(-1)^k n}{p}\right) = -\epsilon$ .

*Proof.* Our assertion is the same as [5, Proposition 1.29] if  $e = 2$  and  $h \in S_{k+1/2}(N, \chi)$ . We can treat the general case on the same line. □

## 2. The space $M_{k+1/2}^+(N, \chi)$

Throughout this paper, we write

$$\mu_{k,\chi} = \chi_2(-1)(-1)^{[(k+1)/2]}, \quad \nu_{k,\chi} = \mathbf{e}((2k+1)(1 - \chi_2(-1))/8).$$

We abbreviate  $\mu = \mu_{k,\chi}$  and  $\nu = \nu_{k,\chi}$  if there is no fear of confusion.

For each integer  $j$ , we put

$$A_j = \begin{pmatrix} 1 & 0 \\ 4Mj & 1 \end{pmatrix}$$

and write  $\mathcal{A} = A_1^* \in \mathfrak{G}$ . For a function  $h$  on  $\mathfrak{H}$ , we define functions  $h|P_4$  and  $h|P_8$  on  $\mathfrak{H}$  by

$$h|P_4 = \sum_{j=0}^3 h|\xi A_j^*, \quad h|P_8 = h|\xi + h|\xi^{-1},$$

where

$$\xi = \left( \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \mathbf{e}((2k+1)(2 - \chi_2(-1))/8) \right) \in \mathfrak{G}.$$

Note that on formal power series

$$(2.1) \quad \sum_{n \in \mathbb{N} \cup \{0\}} a_n q^n | P_8 = 2^{1/2} \mu \left( \sum_{n \in \mathfrak{D}_{k,\chi}} a_n q^n - \sum_{n \notin \mathfrak{D}_{k,\chi}} a_n q^n \right)$$

(cf. [1, (2)]). There are misprints in [1]. We should replace  $\varepsilon$  by  $\chi_2(-1)$ .

**Definition 2.1.** We define the  $\mathbb{C}$ -linear map  $\wp_{k,\chi}$  on formal power series by

$$\sum_{n \in \mathbb{N} \cup \{0\}} a_n q^n | \wp_{k,\chi} = \sum_{n \in \mathfrak{D}_{k,\chi}} a_n q^n.$$

Put  $\wp_k = \wp_{k,\chi}$  if  $\chi$  is the trivial character.

**Proposition 2.2.** (1)  $P_{2^e}$  maps the space  $M_{k+1/2}(N, \chi)$  into itself.

(2) If  $e = 2$ , then we have  $P_4 = P_8 + P_8 \mathcal{A}$  on  $M_{k+1/2}(N, \chi)$ .

(3) If  $e = 3$ , then  $P_8 = 2^{1/2} \mu$  on  $M_{k+1/2}^+(N, \chi)$ , and

$$P_8^2 = 2, \quad \wp_{k,\chi} = 2^{-3/2} \mu P_8 + 2^{-1}$$

on the space  $M_{k+1/2}(N, \chi)$ .

*Proof.* We follow the same line of arguments in [1, pp.36–37]. Put

$$\Delta_0 = \Delta_0(N, \chi) = \left\{ (\gamma, \chi(d)j(\gamma, \tau)^{2k+1}) \mid \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}.$$

For  $\alpha \in \mathfrak{G}$  and  $h \in M_{k+1/2}(N, \chi)$ , we put

$$h | [\Delta_0(N, \chi) \alpha \Delta_0(N, \chi)] = \sum_{\beta \in B_\alpha} h | \beta$$

if there exists a finite set  $B_\alpha \subset \mathfrak{G}$  such that

$$\Delta_0(N, \chi) \alpha \Delta_0(N, \chi) = \bigsqcup_{\beta \in B_\alpha} \Delta_0(N, \chi) \beta.$$

Notice that  $\xi^{-1} \Delta_0(N, \chi) \xi \cap \Delta_0(N, \chi) = \Delta_0(16M, \chi)$  and

$$(2.2) \quad \begin{pmatrix} 1-2M & (M-1)/2 \\ 8M & 1-2M \end{pmatrix}^* \xi \begin{pmatrix} 1 & 0 \\ -8M & 1 \end{pmatrix}^* = \xi^{-1}.$$

Therefore we have

$$B_\xi = \begin{cases} \{\xi, \xi A_1^*, \xi A_2^*, \xi A_3^*\} & \text{if } e = 2. \\ \{\xi, \xi A_2^*\} = \{\xi, \xi^{-1}\} & \text{if } e = 3. \end{cases}$$

It follows that

$$P_{2^e} = [\Delta_0(N, \chi) \xi \Delta_0(N, \chi)]$$

on  $M_{k+1/2}(N, \chi)$ . Thus our assertions are immediate (see (2.1)).  $\square$

**Proposition 2.3.** *Suppose that  $e = 3$ . The operator  $\tilde{Y}(8)$  then maps  $M_{k+1/2}(N, \chi)$  into  $M_{k+1/2}^+(N, \chi)$ , and  $(4\mu\nu)^{-1}\tilde{Y}(8)$  is an involution on  $M_{k+1/2}^+(N, \chi)$ . Moreover, for  $\epsilon \in \{\pm 1\}$  and  $g \in M_{k+1/2}^+(N, \chi)$  the following conditions are equivalent:*

- (i)  $(4\mu\nu)^{-1}g|\tilde{Y}(8) = \epsilon g$ ;
- (ii)  $a_n(g) = 0$  if  $\left(\frac{\chi_2(-1)(-1)^{kn}}{2}\right) = -\epsilon$ .

Here, we put

$$\left(\frac{a}{2}\right) = \begin{cases} 1 & \text{if } a \equiv 1 \pmod{8}. \\ -1 & \text{if } a \equiv 5 \pmod{8}. \\ 0 & \text{if } a \equiv 0, 4 \pmod{8}. \end{cases}$$

*Proof.* Recall that we choose  $\gamma_M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in §1. Let  $h \in M_{k+1/2}(N, \chi)$ . Let  $\eta = -1$  if  $c < 0$  and  $d < 0$ , and let  $\eta = 1$  otherwise. We have

$$h|\tilde{Y}(8) = \eta \operatorname{sgn}(d) \left(\frac{c}{d}\right) \mathbf{e}\left(\frac{2k+1}{8}\right) \sum_{j=0}^7 h|_{k+1/2} \begin{pmatrix} 8(b+jd) & -a-jc \\ 64d & -8c \end{pmatrix}.$$

Since  $b+jd$  and  $8d$  are coprime if  $j$  is an odd integer, we can choose integers  $p, q, r, s$  and  $t$  such that

$$\begin{pmatrix} 8(b+jd) & -a-jc \\ 64d & -8c \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 8 & t \\ 0 & 8 \end{pmatrix}.$$

Note that  $r = 8d$  is divisible by  $N$ . Observing that

$$p \equiv s \equiv -t \equiv j \pmod{8}, \quad s \equiv -c \equiv -1 \pmod{M},$$

we have  $\chi(s) = \chi(-s) = \chi_2(-j)$  and

$$\left(\frac{r}{s}\right) = \left(\frac{8d}{-c-dt}\right) = \eta \operatorname{sgn}(d) \left(\frac{c}{d}\right) \left(\frac{2}{j}\right).$$

Let us set

$$h_i(\tau) = \sum_{\chi_2(-1)(-1)^{kn} \equiv i \pmod{8}} a_n(h) q^n$$

for  $i = 0, \dots, 7$ . Then, using the assumption (II) in §1, we have

$$\begin{aligned} h|\tilde{Y}(8) &= h' + \mathbf{e}\left(\frac{2k+1}{8}\right) \sum_{j \in \{1, 3, 5, 7\}} \chi_2(-j) \epsilon_j^{-2k-1} \left(\frac{2}{j}\right) h|_{k+1/2} \begin{pmatrix} 8 & -j \\ 0 & 8 \end{pmatrix} \\ &= h' + 4\mu\nu(h_1 - h_5), \end{aligned}$$

where we abbreviate the sum over even  $j \in \{0, 2, 4, 6\}$  by  $h'$ .

Observe that  $h'(\tau + 1/2) = h'(\tau)$ . Applying Proposition 1.3 to

$$h|\tilde{Y}(8) - h|\tilde{Y}(8)_{\wp_k} \in M_{k+1/2}(N, \chi),$$



we have  $h'(\tau+1/4) = h'(\tau)$ . Thus  $\tilde{Y}(8)$  maps  $M_{k+1/2}(N, \chi)$  to  $M_{k+1/2}^+(N, \chi)$ .

Replacing  $h$  with  $g$  and repeating the same process, we have

$$g|\tilde{Y}(8)^2 = g'' + (4\mu\nu)^2(g_1 + g_5), \quad g''(\tau + \frac{1}{4}) = g''.$$

The second assertion is an application of Proposition 1.3 to the function  $g|\tilde{Y}(8)^2 - (4\mu\nu)^2g$ . We easily see that (ii) holds if (i) holds. If (ii) holds, then (i) is an application of Proposition 1.3 to  $(4\mu\nu)^{-1}g|\tilde{Y}(8) - \epsilon g$ .  $\square$

**Lemma 2.4.** *If  $e = 3$ , then we have*

$$P_8\mathcal{A}P_8 = 2^{1/2}\mu\mathcal{A}P_8\mathcal{A}$$

on the space  $M_{k+1/2}(N, \chi)$ .

*Proof.* Following the same line of computation as in [1, §2], we have

$$h|\xi A_t^* \xi = \mathbf{e}((2k+1)(\chi_2(-1) - 2)/4)h|_{k+1/2} \begin{pmatrix} 4(1+tM) & 2+tM \\ 16tM & 4(1+tM) \end{pmatrix}$$

for  $h \in M_{k+1/2}(N, \chi)$ , and

$$\begin{pmatrix} 4(1+tM) & 2+tM \\ 16tM & 4(1+tM) \end{pmatrix} = \begin{pmatrix} 1-tM-M^2 & (1+tM)^2/4 \\ -4M^2 & 1+tM+M^2 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} A_t$$

for  $t \in \{\pm 1\}$ . Put  $r_t = 1 + tM + M^2$ . It follows from (2.2) that

$$\begin{aligned} \mathbf{e}\left(\frac{(2k+1)(2-\chi_2(-1))}{8}\right)h|P_8\mathcal{A}P_8 &= \sum_{s, t \in \{\pm 1\}} \chi(r_t)\epsilon_{r_t}^{-2k-1} \left(\frac{-1}{r_t}\right) h|\mathcal{A}\xi A_t^* A_{2s}^* \\ &= \alpha h|\mathcal{A}P_8\mathcal{A}, \end{aligned}$$

where

$$\alpha = \sum_{t \in \{\pm 1\}} \chi_2(t)\epsilon_t^{-2k-1} \left(\frac{-1}{t}\right) = \mathbf{e}(\chi_2(-1)(-1)^k/8)2^{1/2}.$$

We have thus completed the proof of Lemma 2.4.  $\square$

### 3. A certain isomorphism

**Proposition 3.1** (Kohnen). *We have*

$$P_4 = \nu^{-1}\tilde{Y}(4)$$

on the space  $M_{k+1/2}(4M, \chi)$ . Moreover,

$$M_{k+1/2}^+(4M, \chi) = \{h \in M_{k+1/2}(4M, \chi) \mid (2^{3/2}\mu\nu)^{-1}h|\tilde{Y}(4) = h\}$$

and the following direct sum decomposition holds.

$$M_{k+1/2}(4M, \chi) = M_{k+1/2}^+(4M, \chi) \oplus M_{k+1/2}^-(4M, \chi),$$

$$M_{k+1/2}^-(4M, \chi) = \{h \in M_{k+1/2}(4M, \chi) \mid (2^{1/2}\mu\nu)^{-1}h|\tilde{Y}(4) = -h\}.$$

*Proof.* As is well-known,

$$U(4) = 4^{k/2-3/4} [\Delta_0(4M, \chi) \tilde{\delta}_4^{-1} \Delta_0(4M, \chi)]$$

(see the proof of Proposition 2.2 for notation). We have

$$\begin{aligned} \tilde{Y}(4) &= [\Delta_0 \tilde{\delta}_4^{-1} \Delta_0] \tilde{W}(M) \tau(4M) \\ &= \left[ \Delta_0 \left( \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix}, 4^{k/2+1/4} \right) \Delta_0 \right] \tilde{W}(M) \tau(4M) \\ &= \left[ \Delta_0 \left( \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix}, 4^{k/2+1/4} \right) \tilde{W}(M) \tau(4M) \Delta_0 \right] \\ &= \left[ \Delta_0 \left( \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \mathbf{e}((2k+1)/8) \right) \Delta_0 \right] = \nu P_4 \end{aligned}$$

as in the proof of Proposition 2.3. We know that the similar statement holds for the space  $S_{k+1/2}(4M, \chi)$ , taking [1, Proposition 1] into account. We can treat  $M_{k+1/2}(4M, \chi)$  by the same way.  $\square$

*Remark 3.2.* In a similar method, we can show that

$$\nu^{-1} \tilde{Y}(8) = [\Delta_0(8M, \chi) \eta \Delta_0(8M, \chi)],$$

where

$$\eta = \left( \begin{pmatrix} 8 & 1 \\ 0 & 8 \end{pmatrix}, \mathbf{e}((2k+1)(2 - \chi_2(-1))/8) \right).$$

The following proposition is important since at the moment, our knowledge of the space  $S_{k+1/2}(8M, \chi)$  is incomplete, although there are a number of results available in the space  $S_{k+1/2}(4M, \chi)$ . For example, fairly good information on  $S_{k+1/2}(4M, \chi)$  was obtained in [6].

**Proposition 3.3.** *The map  $\wp_{k,\chi}$  induces a  $\mathbb{C}$ -linear isomorphism of  $M_{k+1/2}(4M, \chi)$  onto  $M_{k+1/2}^+(8M, \chi)$ , which maps  $S_{k+1/2}(4M, \chi)$  onto  $S_{k+1/2}^+(8M, \chi)$ .*

*Proof.* Put

$$\varrho_{k,\chi} = 2^{-1}(1 + \mathcal{A})(3 - 2^{-1/2} \mu P_4).$$

The map  $\varrho_{k,\chi}$ , which sends  $M_{k+1/2}^+(8M, \chi)$  into  $M_{k+1/2}(4M, \chi)$ , turns out to be the inverse of  $\wp_{k,\chi}$ . Indeed, Propositions 2.2 (2) and 3.1 show that

$$\wp_{k,\chi} \varrho_{k,\chi} = (2^{-3/2} \mu P_8 + 2^{-1}) \varrho_{k,\chi} = 2^{-1} (2^{-3/2} \mu P_4 + 1) (3 - 2^{-1/2} \mu P_4) = 1$$

on  $M_{k+1/2}(4M, \chi)$ . It follows that

$$\begin{aligned} (3 - 2^{-1/2} \mu P_4) \wp_{k,\chi} &= (3 - 2^{-1/2} \mu (P_8 + P_8 \mathcal{A})) (2^{-3/2} \mu P_8 + 2^{-1}) \\ &= 1 + 2^{-1/2} \mu P_8 - 2^{-3/2} \mu P_8 \mathcal{A} - 2^{-2} P_8 \mathcal{A} P_8. \end{aligned}$$

By virtue of Lemma 2.4, we have

$$\varrho_{k,\chi} \wp_{k,\chi} = (1 + 2^{-1/2} \mu P_8 - (2^{-1/2} \mu P_8 - 1) \mathcal{A} (1 + 2^{-1/2} \mu P_8)) / 2.$$

Since  $P_8 = 2^{1/2}\mu$  on  $M_{k+1/2}^+(8M, \chi)$ , we conclude that  $\varrho_{k,\chi}\varphi_{k,\chi} = 1$ .  $\square$

*Corollary 3.4.* Let  $g \in S_{k+1/2}(4M, \chi)$  and  $h \in M_{k+1/2}(4M, \chi)$ . Write

$$\begin{aligned} g &= g_1 + g_2, & h &= h_1 + h_2, \\ g_1, h_1 &\in M_{k+1/2}^+(4M, \chi), & g_2, h_2 &\in M_{k+1/2}^-(4M, \chi) \end{aligned}$$

(see Proposition 3.1 for the definition of  $M_{k+1/2}^-(4M, \chi)$ ). Then we have

$$\langle g|_{\varphi_{k,\chi}}, h|_{\varphi_{k,\chi}} \rangle = \langle g_1, h_1 \rangle + \frac{\langle g_2, h_2 \rangle}{4}.$$

*Proof.* Observing Propositions 2.2, 3.3 and their proofs, we have

$$\begin{aligned} \langle g|_{\varphi_{k,\chi}}, h|_{\varphi_{k,\chi}} \rangle &= \langle g, h|_{\varphi_{k,\chi}^2} \rangle = \langle g, h|_{\varphi_{k,\chi}} \rangle \\ &= \langle g, h|_{\varphi_{k,\chi}(1 + \mathcal{A})} \rangle / 2 = \langle g, h|(2^{-3/2}\mu P_4 + 1) \rangle / 2. \end{aligned}$$

Corollary 3.4 is an easy consequence of Proposition 3.1.  $\square$

We shall use Corollaries 3.4 and 5.2 in a forthcoming paper.

#### 4. Results of Waldspurger

We use the notation in [6] in this section. Let  $\phi \in S_{2k}(\Gamma_0(d))$  be a primitive form. Put  $\lambda_p = p^{-k+1/2}a_p(\phi)$  for each prime number  $p$ . Choose a complex number  $\alpha_p$  such that  $\alpha_p + \alpha_p^{-1} = \lambda_p$  when  $p$  and  $d$  are coprime. We denote by  $\rho_v$  the local component at a place  $v$  of the automorphic representation determined by  $\phi$ . Let  $S$  be the set of places such that  $\rho_v$  is not a irreducible principal series.

We assume the following conditions:

- (H<sub>1</sub>) If  $p \notin S$ , then  $\rho_p \simeq \pi(\mu_p, \mu_p^{-1})$  with  $\mu_p(-1) = 1$ ;
- (H<sub>2</sub>)  $\rho_2$  is not supercuspidal or  $d$  is divisible by 16.

Let  $V(N)$  be the orthogonal complement of the space spanned by theta functions in  $S_{3/2}(N)$ . The space  $S_{k+1/2}(N, \phi)$  consists of all functions  $g \in S_{k+1/2}(N)$  (resp.  $V(N)$ ) such that  $g|\tilde{T}(p^2) = a_p(\phi)g$  for every rational prime  $p \nmid N$  if  $k > 1$  (resp.  $k = 1$ ).

For each rational prime  $p$  and non-negative integer  $e$ , Waldspurger [6] associated to  $\phi$  a non-negative integer  $\tilde{n}_p$  and a set  $U_p(e, \phi)$  of functions on  $\mathbb{Q}_p^\times$  invariant under  $\mathbb{Z}_p^{\times 2}$ , the supports of which lie within  $\mathbb{Z}_p \cap \mathbb{Q}_p^\times$ . Put  $\tilde{N}(\phi) = \prod_p p^{\tilde{n}_p}$ .

*Remark 4.1.* If  $p$  and  $2d$  are coprime, then  $U_p(0, \phi) = \{c_p^0[\lambda_p]\}$ , where we define  $c_p^0[\lambda_p]$  in the following way. We put

$$l_e(X) = \begin{cases} \frac{X^{e+1} - X^{-e-1}}{X - X^{-1}} & \text{if } e \geq 0 \\ 0 & \text{if } e < 0 \end{cases}$$

for each integer  $e$ . For each rational prime  $p$  and  $a \in \mathbb{Q}_p^\times$ , we put

$$\lambda_{p,a} = l_{\mathfrak{f}_p(a)} - \underline{\psi}_p((-1)^k a) p^{-1/2} l_{\mathfrak{f}_p(a)-1}$$

and

$$\mathfrak{f}_p(a) = \left\lfloor \frac{\text{ord}_p a + 1 - \delta_{p2}}{2} \right\rfloor - 1 + \underline{\psi}_p((-1)^k a)^2,$$

where  $\delta_{ij}$  be the Kronecker delta, and  $\underline{\psi}_p(a) = 1, -1, 0$  accordingly as  $\mathbb{Q}_p(\sqrt{a})$  is  $\mathbb{Q}_p$ , an unramified quadratic extension of  $\mathbb{Q}_p$  or a ramified quadratic extension of  $\mathbb{Q}_p$ . Then we put  $c_p^0[\lambda_p](a) = \lambda_{p,a}(\alpha_p)$  for every  $a \in \mathbb{Q}_p^\times$ .

Let  $\mathbb{N}^{\text{sc}}$  be the set of positive square-free integers. For each positive integer  $n$ , the unique element of the set  $\mathbb{N}^{\text{sc}} \cap n\mathbb{Q}^{\times 2}$  is denoted by  $n^{\text{sc}}$ . Let  $\underline{A}$  be a function on  $\mathbb{N}^{\text{sc}}$  and  $E$  a positive integer which is divisible by  $\tilde{N}(\phi)$ . Put  $e_p = \text{ord}_p E$  and define a function  $g(\underline{c}_E, \underline{A})$  on  $\mathfrak{H}$  by

$$g(\underline{c}_E, \underline{A})(\tau) = \sum_{n \in \mathbb{N}} \underline{A}(n^{\text{sc}}) n^{k/2-1/4} \prod_p c_p(n) q^n$$

for  $\underline{c}_E = (c_p) \in \prod_p U_p(e_p, \phi)$ . We write  $\overline{U}(E, \phi, \underline{A})$  for the space spanned by functions  $g(\underline{c}_E, \underline{A})$  for  $\underline{c}_E \in \prod_p U_p(e_p, \phi)$ .

Waldspurger showed the following results in slightly more general situations.

**Theorem 4.2** (cf. [6, Theorem 1]). *Suppose that  $\phi$  satisfies the hypotheses  $(H_1)$  and  $(H_2)$ . There exists a function  $\underline{A}^\phi : \mathbb{N}^{\text{sc}} \rightarrow \mathbb{C}$  which satisfies the following conditions:*

(i) *for every  $t \in \mathbb{N}^{\text{sc}}$*

$$\underline{A}^\phi(t)^2 = L(\phi\psi_{(-1)^k t}, 1/2) \varepsilon(\psi_{(-1)^k t}, 1/2),$$

*where  $L(\phi\psi_{(-1)^k t}, 1/2)$  is the central critical value of the  $L$ -function of  $\phi$  twisted with the quadratic character corresponding to  $\mathbb{Q}(\sqrt{(-1)^k t})$ ;*

(ii) *for every positive integer  $N$*

$$S_{k+1/2}(N, \phi) = \oplus_{\tilde{N}(\phi) | E | N} \overline{U}(E, \phi, \underline{A}^\phi),$$

*where  $E$  extends over all positive divisors of  $N$  divisible by  $\tilde{N}(\phi)$ .*

**Proposition 4.3.** *Assume that  $d$  is square-free. Put  $t_p = \log \alpha_p / \log p$  or  $\log \lambda_p / \log p$  accordingly as  $p \notin S$  or  $p \in S$ . Then the following assertions hold.*

- (1)  *$S$  consists of the infinite place and all prime divisors of  $d$ .*
- (2) *If  $p \notin S$ ,  $\rho_p$  is the principal series  $\pi(| \cdot |_p^{-t_p}, | \cdot |_p^{t_p})$ .*

TABLE 1

	$U_p(0, \phi)$	$U_p(1, \phi)$	$U_p(2, \phi)$
$2 \neq p \notin S$	$\{c_p^0[\lambda_p]\}$	$\{c_p'[\alpha_p]\}$	
$2 \neq p \in S$	$\emptyset$	$\{c_p^s[\lambda_p]\}$	
$2 = p \notin S, \alpha_2 \neq \pm 1$	$\emptyset$	$\emptyset$	$\{c_2'[\alpha_2], c_2'[\alpha_2^{-1}]\}$
$2 = p \notin S, \alpha_2 \in \{\pm 1\}$	$\emptyset$	$\emptyset$	$\{c_2'[\alpha_2], c_2''[\alpha_2]\}$
$2 = p \in S$	$\emptyset$	$\emptyset$	$\{c_2^s[\lambda_2]\}$

- (3) If  $p$  is a rational prime in  $S$ , then  $\lambda_p$  is either  $p^{-1/2}$  or  $-p^{-1/2}$ , and  $\rho_p$  is the special representation  $\sigma(|\cdot|_p^{-t_p}, |\cdot|_p^{t_p})$ .
- (4)  $S_{3/2}(4d) = V(4d)$ .
- (5)  $\tilde{n}_p = \text{ord}_p d$  or  $2$  accordingly as  $p$  is odd or  $p = 2$ .
- (6) Assume that  $N$  is divisible by  $d$  and  $4$ . Then all functions  $g \in S_{k+1/2}(N, \phi)$  satisfies the following conditions for each prime number  $p \in S$ :
  - (a) If  $\psi_p((-1)^k n) = p^{1/2} \lambda_p$ , then  $a_n(g) = 0$ ;
  - (b)  $g|U(p^2) = p^{k-1/2} \lambda_p g$  if  $(2p)^{-1}N$  and  $p$  are coprime.

*Remark 4.4.* From Proposition 4.3 (2) and (3), the hypotheses  $(H_1)$  and  $(H_2)$  are automatically satisfied if  $d$  is square-free.

*Proof.* Our assertions (1), (2) and (3) are well-known. See [4, §3 Corollary] for (4). Our assertion (5) directly follows from the definition of  $\tilde{n}_p$ . Our assertion (a) is a special case of [6, Proposition 19].

The set  $U_p(e, \phi)$  specifies as in Table 1. For each  $\delta \in \mathbb{C}$  and odd rational prime  $p$ , the definitions of functions  $c_p'[\delta]$ ,  $c_p^s[\delta]$ ,  $c_2'[\delta]$ ,  $c_2''[\delta]$  and  $c_2^s[\delta]$  can be found in [6] (for  $c_p^0[\delta]$  see remark 4.1). Let us note that  $c_p^s[\lambda_p](ap^2) = \lambda_p c_p^s[\lambda_p](a)$  for every  $a \in \mathbb{Z}_p \cap \mathbb{Q}_p^\times$ . We thus obtain the assertion (b) in view of Theorem 4.2 and Table 1.  $\square$

If  $d$  is square-free, then we define a function  $g^\phi$  on  $\mathfrak{H}$  by

$$g^\phi(\tau) = \sum_{n \in \mathbb{N}} \underline{A}^\phi(n^{\text{sc}}) n^{k/2-1/4} \prod_{p \in S} c_p^s[\lambda_p](n) \prod_{p \notin S} \lambda_{p,n}(\alpha_p) q^n$$

(see Remark 4.1 and the proof of Proposition 4.3 for  $c_p^s[\lambda_p]$  and  $\lambda_{p,n}$ ).

**Proposition 4.5.** *If  $2M$  is square-free and  $d$  is a positive divisor of  $2M$ , then*

$$S_{k+1/2}(4M, \phi) = \oplus_{b|2d^{-1}M} \mathbb{C} g^\phi | U(b^2),$$

where  $b$  extends over all positive divisors of  $2d^{-1}M$ .

*Proof.* If  $d$  is odd, then we can easily check that

$$\lambda_{2,a}(\alpha_2) = \begin{cases} \frac{\alpha_2 c'_2[\alpha_2](a) - \alpha_2^{-1} c'_2[\alpha_2^{-1}](a)}{\alpha_2 - \alpha_2^{-1}} & \text{if } \alpha_2 \neq \pm 1 \\ \alpha_2 c''_2[\alpha_2](a) & \text{if } \alpha_2 \in \{\pm 1\} \end{cases}$$

for every  $a \in \mathbb{Q}_2^\times$ . Remarks 4.1, 4.4 and Theorem 4.2 thus show that  $g^\phi \in S_{k+1/2}(\tilde{N}(\phi), \phi)$ . It follows from an easy computation that

$$c'_p[\alpha_p](a) = (\lambda_{p,p^2a}(\alpha_p) - \alpha_p^{-1} \lambda_{p,a}(\alpha_p)) \times \begin{cases} \alpha_p^{-1} & \text{if } p \neq 2 \\ 1 & \text{if } p = 2 \end{cases}$$

for every  $p \notin S$  and  $a \in \mathbb{Q}_p^\times$ . Theorem 4.2, Proposition 4.3 (5) and Table 1 show that  $S_{k+1/2}(4M, \phi)$  is spanned by functions  $g^\phi|U(b^2)$  with positive divisors  $b$  of  $2d^{-1}M$ .

We have only to show that these functions are linearly independent. For a positive integer  $N$ , the symbol  $t(N)$  stands for the number of prime divisors of  $N$ . We see that

$$(4.1) \quad \dim_{\mathbb{C}} S_{k+1/2}(4M, \phi) \leq 2^{t(2d^{-1}M)}.$$

In view of Proposition 4.3 (4), we know that

$$(4.2) \quad S_{k+1/2}(4M) = \bigoplus_{d|2M, \phi \in \text{Prm}_{2k}(d)} S_{k+1/2}(4M, \phi),$$

where  $\text{Prm}_{2k}(d)$  denotes the set of primitive forms in  $S_{2k}(\Gamma_0(d))$ . Since

$$S_{2k}(\Gamma_0(2M)) = \bigoplus_{b, d \geq 1, bd|2M, \phi \in \text{Prm}_{2k}(d)} \mathbb{C} \phi|_{2k} \begin{pmatrix} b & \\ & 1 \end{pmatrix},$$

it follows that

$$(4.3) \quad \dim_{\mathbb{C}} S_{2k}(\Gamma_0(2M)) = \sum_{d|2M} 2^{t(2d^{-1}M)} \#\text{Prm}_{2k}(d).$$

Since [4, §3 Corollary] shows that

$$\dim_{\mathbb{C}} S_{2k}(\Gamma_0(2M)) = \dim_{\mathbb{C}} S_{k+1/2}(4M),$$

(4.2) and (4.3) show that the equality holds in (4.1). The proof of Proposition 4.5 is now complete.  $\square$

## 5. Proof of Main Theorem

We now prove Theorem (see Introduction for notation and the statement). If  $e = 3$ , then Propositions 3.3, 4.5 and (4.2) show that

$$(5.1) \quad \begin{aligned} S_{k+1/2}^+(N) &= S_{k+1/2}(2^{-1}N)|_{\wp_k} = \bigoplus_{d|4^{-1}N, \phi \in \text{Prm}_{2k}(d)} S_{k+1/2}(2^{-1}N, \phi)|_{\wp_k} \\ &= \bigoplus_{b, d \geq 1, bd|4^{-1}N, \phi \in \text{Prm}_{2k}(d)} \mathbb{C} g^\phi|_{\wp_k} U_k(b^2). \end{aligned}$$

Note that (5.1) also holds when  $e = 2$ . It follows that

$$(5.2) \quad S_{k+1/2}^{\text{new},+}(N) = \bigoplus_{\phi \in \text{Prm}_{2k}(4^{-1}N)} \mathbb{C}g^\phi|_{\wp_k}.$$

Proposition 4.3 (b) shows that  $g^\phi|_{\wp_k}U_k(q^2) = \omega_q g^\phi|_{\wp_k}$  for every prime divisor  $q$  of  $4^{-1}N$ . We thus observe that functions  $g^\phi|_{\wp_k}$  satisfy the conditions of (3). We are led to our assertion (1) and (2) by virtue of (5.1) and (5.2).

*Remark 5.1.* We can establish the newform theory for an even quadratic character  $\chi$ . Letting  $\mathfrak{f}$  be the conductor of  $\chi$  and observing that  $M_{k+1/2}^+(N)|U(\mathfrak{f}') = M_{k+1/2}^+(N, \chi)$ , where  $\mathfrak{f}'$  is either  $\mathfrak{f}$  or  $\mathfrak{f}/4$  according as  $\mathfrak{f}$  is odd or even, we can define the space of newforms for  $S_{k+1/2}^+(N, \chi)$  by

$$S_{k+1/2}^{\text{new},+}(N, \chi) = S_{k+1/2}^{\text{new},+}(N)|U(\mathfrak{f}'),$$

and can generalize our result as in [1].

*Corollary 5.2.* Under the notation as in Theorem , the following assertions holds.

- (1) On the space  $S_{k+1/2}^{\text{new},+}(N)$ , we have

$$p^{1-k}U_k(p^2) = \begin{cases} -\epsilon_p^{2k+1}p^{-1/2}\tilde{Y}(p) & \text{if } 2 \neq p|4^{-1}N. \\ -(4\mu\nu)^{-1}\tilde{Y}(8) & \text{if } 2 = p|4^{-1}N. \end{cases}$$

- (2) Let  $p$  be a prime divisor of  $4^{-1}N$ . If  $g \in S_{k+1/2}^+(p^{-1}N)$  is an eigenfunction of the operator  $\tilde{Y}(p^{\text{ord}_p N})$ , then  $g = 0$ .

*Proof.* Combining Propositions 1.4, 2.3, 4.3 (6) with (5.2), we can establish our assertion (1).

If  $p$  is odd, then (1.1) and the definition of  $\tilde{Y}(p)$  show that  $g|U(p)$  and  $g|\tilde{W}(p) = g|\tilde{\delta}_p$  are linearly dependent. This contradicts Theorem (1) for  $g|\tilde{\delta}_p U(p) = p^{k/2+1/4}g$ . If  $p = 2$ , then taking Proposition 3.1 into account, we can show that  $g|U(4)\tilde{\delta}_2$  and  $g|U(8)$  are linearly dependent, which contradicts Theorem (1).  $\square$

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